A CORRESPONDENCE BETWEEN REPRESENTATIONS OF LOCAL GALOIS GROUPS AND LIE-TYPE GROUPS

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Serre conjectured in [13] that every continuous, irreducible odd representation

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

arises from a modular form. Moreover he refines the conjecture by specifying an optimal weight and level for a Hecke eigenform giving rise to ρ. Viewing Serre's conjecture as a manifestation of Langlands' philosophy in characteristic p, this refinement can be viewed as a local-global compatibility principle, the weight of the form reflecting the behavior of ρ at p, the level reflecting the behavior at primes other than p. The equivalence between the "weak" conjecture and its refinement (for $\ell > 2$) follows from work of Ribet [11] and others (see [6]). Remarkable progress has recently been made on the conjecture itself by Khare and Wintenberger; see for example Khare's article in this volume.

Serre's conjecture is generalized in [5] to the context of Hilbert modular forms and two-dimensional representations of G_K where K is a totally real number field in which p is unramified. The difficulty in formulating the refinement lies in the specification of the weight. This is handled in [5] by giving a recipe for a set $W_{\mathfrak{p}}(\rho)$ of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathcal{O}_K/\mathfrak{p})$ for each prime $\mathfrak{p}|p$ in terms of $\rho|_{I_{\mathfrak{p}}}$; the sets $W_{\mathfrak{p}}(\rho)$ then conjecturally characterize the types of local factors at primes over p of automorphic representations giving rise to ρ. We omit the subscript $\mathfrak{p}$ since we shall be concerned only with local behavior, so now K will denote a finite unramified extension of $\mathbb{Q}_p$ with residue field k.

The purpose of the paper is to prove that if the local Galois representation is semisimple, then $W(\rho)$ is essentially the set of Jordan-Hölder constituents of the reduction of an irreducible *characteristic zero* representation of $\mathrm{GL}_2(k)$. Moreover, denoting this representation $\alpha(\rho)$ we obtain

Theorem 0.1. *There is a bijection*

$$\{\rho : G_K \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}}_p)\}/\sim \ \textit{ of } \rho|_{I_K}^{\mathrm{ss}}$$

$$\alpha \updownarrow$$

$$\{\textit{irreducible } \overline{\mathbb{Q}}_p\textit{-representations of } \mathrm{GL}_2(k) \textit{ not factoring through } \det\}/\sim$$

such that $W(\rho^{\mathrm{ss}})$ *contains the set of Jordan-Hölder factors of the reduction of* $\alpha(\rho)$.

Moreover, the last inclusion is typically an equality and one can explicitly describe the exceptional weights. We remark that the local Langlands correspondence also gives rise to a bijection between the sets in the theorem by taking the K-type

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corresponding to a tamely ramified lift of ρ. The bijection of the theorem however has a different flavor. Indeed if $[k:\mathbb{F}_p]$ is odd, then irreducible ρ correspond to principal series and special representations, while reducible ρ correspond to supercuspidal ones.

A generalization of Serre's Conjecture to the setting of GL_n was formulated by Ash and others in [1], [2], and Herzig's thesis [8] pursues the idea of relating the set of Serre weights of a semi-simple $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ to the reduction of an irreducible characteristic zero representation of $\mathrm{GL}_n(\mathbb{F}_p)$. However Herzig shows that the phenomenon described in Theorem 0.1 does *not* persist for $n > 2$; instead he defines an operator $\mathcal{R}$ on the irreducible mod p representations of $\mathrm{GL}_n(\mathbb{F}_p)$ and shows that the regular (i.e., up to certain exceptions) Serre weights of ρ are given by applying $\mathcal{R}$ to the constituents of the reduction of a certain characteristic zero representation $V(\rho)$. Herzig also show that such a relationship holds in the context of $\mathrm{GL}_2(k)$. Moreover, the association $\rho \mapsto V(\rho)$ appears to be compatible with the local Langlands correspondence in the sense described above. In this light, Theorem 0.1 can be viewed as saying that Herzig's operator $\mathcal{R}$ typically sends the set of irreducible constituents of the reduction of one $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_2(k)$ to those of another.

One can also view Theorem 0.1 in the context of the theory of mod p and p-adic local Langlands correspondences being developed by Breuil and others (see [3], [4], [7]). In particular, one would like a mod p local Langlands correspondence to associate a mod p representation of $\mathrm{GL}_2(K)$ to ρ, and local-global compatibility considerations suggest that the set of Serre weights comprise the constituents of its $\mathrm{GL}_2(\mathcal{O}_K)$-socle. One would also like a p-adic local Langlands correspondence associating p-adic representations of $\mathrm{GL}_2(K)$ to suitable lifts of ρ, and satisfying some compatibility with the mod p correspondence with respect to reduction. One can thus speculate that the theorem reflects some property of the hypothetical p-adic correspondence for GL_2.

The paper is organized as follows: In Section 1, we compute the semisimplification of the reduction mod p of the irreducible characteristic zero representations of $\mathrm{GL}_2(k)$. The main theorem is proved in Section 2, and the exceptional weights are described in Section 3 for the sake of completeness.

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1. A BRAUER CHARACTER COMPUTATION

We first recall the irreducible $\overline{\mathbb{Q}}_p$-representations of $G = \mathrm{GL}_2(k)$ (see for example [9, Ch. 28] or [10, XVIII, §12]).

Let B denote the subgroup of upper-triangular matrices in G. For a pair of homomorphisms $\chi_1, \chi_2 : k^\times \to \overline{\mathbb{Q}}^\times$, we let $I(\chi_1, \chi_2)$ denote the $q+1$-dimensional representation of G induced from the character of B defined by

$$\begin{pmatrix} x & w \\ 0 & y \end{pmatrix} \mapsto \chi_1(x)\chi_2(y).$$

$I(\chi_1, \chi_2) \sim I(\chi_1', \chi_2')$ if and only if $\{\chi_1, \chi_2\} = \{\chi_1', \chi_2'\}$. If $\chi_1 \neq \chi_2$, then $I(\chi_1, \chi_2)$ is irreducible. $I(\chi, \chi) \sim \chi \circ \det \oplus \mathrm{sp}_\chi$ for an irreducible q-dimensional representation sp_χ.

The remaining irreducible $\overline{\mathbb{Q}}$-representations of G are parametrized as follows. Let k' be a quadratic extension of k, σ the non-trivial k-automorphism of k' and Nm the norm from k' to k. For each homomorphism $\xi : k'^{\times} \to \overline{\mathbb{Q}}^{\times}$ such that $\xi \neq \xi \circ \sigma$, there is an irreducible $(q-1)$-dimensional $\overline{\mathbb{Q}}$-representation $\Theta(\xi)$ of G, and $\Theta(\xi) \sim \Theta(\xi')$ if and only if $\xi' \in \{\xi, \xi \circ \sigma\}$. Moreover for any homomorphism $\chi : k^{\times} \to \overline{\mathbb{Q}}^{\times}$, we have $(\chi \circ \det)\Theta(\xi) \sim \Theta((\chi \circ \mathrm{Nm})\xi))$.

Letting i denote a k-algebra embedding $k' \to \mathrm{M}_2(k)$, the character table of G is as follows:

Conjugacy class of:	Representation			
	$\chi \circ \det$	sp_{χ}	$I(\chi_1, \chi_2)$	$\Theta(\xi)$
$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	$\chi(x)^2$	$q\chi(x)^2$	$(q+1)\chi_1(x)\chi_2(x)$	$(q-1)\xi(x)$
$\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$\chi(x)^2$	0	$\chi_1(x)\chi_2(x)$	$-\xi(x)$
$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \notin k^{\times}$	$\chi(xy)$	$\chi(xy)$	$\chi_1(x)\chi_2(y) + \chi_1(y)\chi_2(x)$	0
$i(z) \notin k^{\times}$	$\chi(zz^{\sigma})$	$-\chi(zz^{\sigma})$	0	$-\xi(z) - \xi(z^{\sigma})$

Next we recall the irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$. Let $S = k(\overline{\mathbb{F}}_p)$, the set of embeddings $k \to \overline{\mathbb{F}}_p$. For integers m_τ, n_τ with $n_\tau \geq 0$ for each $\tau \in S$, we have the representation

$$V_{\vec{m},\vec{n}} = \otimes_{\tau \in S} \det{}^{m_\tau} k^2 \otimes_k \mathrm{Sym}^{n_\tau - 1} k^2 \otimes_{k,\tau} \overline{\mathbb{F}}_p.$$

We make the convention that $\mathrm{Sym}^{-1} = 0$, so that $V_{\vec{m},\vec{n}}$ has dimension $\prod_{\tau \in S} n_\tau$. If $1 \leq n_\tau \leq p$ for all τ, then $V_{\vec{m},\vec{n}}$ is irreducible; assuming further that $0 \leq m_\tau \leq p-1$ for each τ and some $m_\tau < p-1$, then the $V_{\vec{m},\vec{n}}$ are inequivalent and form a complete list of the irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$.

Recall that the semisimplification of an $\overline{\mathbb{F}}_p$-representation of G is determined by its Brauer character, which is a $\overline{\mathbb{Q}}_p$-valued function on the p-regular conjugacy classes of G (see [12, 18.1, 18.2] for example). Letting $\tilde{}$ denote the Teichmüller lift, the Brauer character of $V_{\vec{m},\vec{n}}$, which we denote $\beta_{\vec{m},\vec{n}}$, is as follows:

$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$	$\prod_{\tau \in S} \left(\tilde{\tau}(xy)^{m_\tau} \sum_{0 \leq \nu \leq n_\tau - 1} \tilde{\tau}(y)^{\nu} \tilde{\tau}(x)^{n_\tau - 1 - \nu} \right)$
$i(z) \notin k^{\times}$	$\prod_{\tau \in S} \left(\tilde{\tau}'(z)^{(q+1)m_\tau} \sum_{0 \leq \nu \leq n_\tau - 1} \tilde{\tau}'(z)^{n_\tau - 1 + (q-1)\nu} \right)$

where τ' denotes either extension of τ to k'.

If V is a finite-dimensional $\overline{\mathbb{Q}}_p$-representation of G, then there exists a $\overline{\mathbb{Z}}_p$-lattice $L \subset V$ stable under the action of G. Reducing L modulo the maximal ideal of $\overline{\mathbb{Z}}_p$ then yields an $\overline{\mathbb{F}}_p$-representation $\overline{L}$ of G whose Brauer character is the restriction of the character of V to the p-regular classes of G. In particular, the semisimplification of $\overline{L}$ is independent of the choice of lattice L, and we denote it $\overline{V}$ and call it the *reduction* of V.

We now compute $\overline{V}$ for all irreducible V (i.e., the decomposition matrix of G with respect to reduction mod p). First note that any homomorphism $\chi : k^{\times} \to \overline{\mathbb{Q}}_p^{\times}$

can be written in the form $\prod_\tau \tilde\tau^{a_\tau}$ for some integers a_τ with $0 \le a_\tau \le p-1$, in which case $\overline{\chi} = \prod_\tau \tau^{a_\tau}$. Moreover, if $V' \sim (\chi \circ \det) \otimes V$, then $\overline{V}' \sim (\overline{\chi} \circ \det) \otimes \overline{V}$, so we can replace V by such a twist in order to compute its reduction.

We first consider the representations $I(\chi_1, \chi_2)$. Twisting by $\chi_1^{-1} \circ \det$, we need only consider those of the form $I(1, \chi)$. The reduction is then given by the following proposition:

Proposition 1.1. *Let $V = I(1, \prod_\tau \tilde\tau^{a_\tau})$ with $0 \le a_\tau \le p-1$ for each $\tau \in S$. Then $\overline{V} \sim \oplus_{J \subset S} V_J$, where $V_J = V_{\vec m_J, \vec n_J}$ with $\vec m_J$ and $\vec n_J$ defined as follows:*

$$\begin{aligned} m_{J,\tau} &= \begin{cases} 0, & \text{if } \tau \in J, \\ a_\tau + \delta_J(\tau), & \text{if } \tau \notin J, \end{cases} \\ \text{and} \quad n_{J,\tau} &= \begin{cases} a_\tau + \delta_J(\tau), & \text{if } \tau \in J, \\ p - a_\tau - \delta_J(\tau), & \text{if } \tau \notin J, \end{cases} \end{aligned}$$

where δ_J is the characteristic function of $J^{(p)} = \{\, \tau \circ \mathrm{Frob} \mid \tau \in J \,\}$. Moreover the non-zero V_J are inequivalent.

Proof. We need to show that the sum of the $\beta_{\vec m_J, \vec n_J}$ coincides with the character of V on p-regular conjugacy classes.

We first consider conjugacy classes of elements of the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with $x, y \in k$. Let us choose an embedding $\tau_0 : k \to \overline{\mathbb{F}}_p$ and index the elements of S by setting $\tau_i = \tau \circ \mathrm{Frob}_p^i$ for $i \in \mathbb{Z}/f\mathbb{Z}$. We then have

$$\begin{aligned} \beta_{\vec m, \vec n}\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) &= \tilde x^{\sum_{i=0}^{f-1} m_i p^i} \tilde y^{\sum_{i=0}^{f-1} m_i p^i} \sum_{\vec 0 \le \vec \nu \le \vec n - \vec 1} \tilde y^{\sum_{i=0}^{f-1} \nu_i p^i} \tilde x^{\sum_{i=0}^{f-1} (n_i - 1 - \nu_i) p^i} \\ &= \tilde x^{\sum_{i=0}^{f-1} (2m_i + n_i - 1) p^i} \sum_{\vec m \le \vec b \le \vec m + \vec n - \vec 1} (\tilde y / \tilde x)^{\sum_{i=0}^{f-1} b_i p^i}, \end{aligned}$$

where we have simply written m_i for m_{τ_i}, n_i for n_{τ_i} and $\tilde w$ for $\tilde\tau_0(w)$. (We also abuse notation in viewing i as an integer when it appears as an exponent of p and as a congruence class when it appears as an index.) Taking $(\vec m, \vec n) = (\vec m_J, \vec n_J)$ and viewing $J \subset \{0, 1, \ldots, f-1\}$, we have

(1)
$$\begin{aligned} \textstyle\sum_{i=0}^{f-1} (2m_i + n_i - 1) p^i &= \textstyle\sum_{i \in J} (a_i - 1 + \delta_J(i)) p^i + \sum_{i \notin J} (a_i - 1 + \delta_J(i) + p) p^i \\ &= (1 - \delta_J(0))(q-1) + \textstyle\sum_{i=0}^{f-1} a_i p^i, \end{aligned}$$

so that $\tilde x^{\sum_{i=0}^{f-1} (2m_i + n_i - 1) p^i} = \tilde x^{\sum_{i=0}^{f-1} a_i p^i}$, giving

$$\beta_{\vec m_J, \vec n_J}\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \tilde x^{\sum_{i=0}^{f-1} a_i p^i} \left(\sum_{d \in B_J} (\tilde y / \tilde x)^d \right)$$

where

$$B_J = \left\{ d = \sum_{i=0}^{f-1} b_i p^i \;\middle|\; \begin{array}{ll} 0 \le b_i < a_i + \delta_J(i), & \text{if } i \in J, \\ a_i + \delta_J(i) \le b_i < p, & \text{if } i \notin J \end{array} \right\}. \tag{2}$$

Note that if $d \in B_J$, then $0 \le d \le q-1$. Since the only dependence relation among the functions $w \mapsto \tilde w^d$ on $k^\times$ for $0 \le d \le q-1$ is that $\tilde w^0 = \tilde w^{q-1}$, we see that $V_J = 0$ if and only if $B_J = \emptyset$. Moreover if $V_J \sim V_{J'}$, then either $B_J = B_{J'}$ or one is gotten from the other by replacing 0 by $q-1$. One sees easily that the first case implies that either $B_J = \emptyset$ or $J = J'$, and that the second is impossible. We thus conclude that the the non-zero $V_{\vec{m}_J,\vec{n}_J}$ are inequivalent.

To complete the proof of the proposition, we use another description of the B_J:

Lemma 1.2. *Suppose that* $0 \le d \le q-1$. *Write* $d = \sum_{i=0}^{f-1} b_i p^i$ *with* $0 \le b_i \le p-1$ *for each* $i \in \mathbb{Z}/f\mathbb{Z}$. *If* $d \ne \sum_{i=0}^{f-1} a_i p^i$, *then*

$$d \in B_J \iff J = \left\{ j \in \mathbb{Z}/f\mathbb{Z} \;\middle|\; \sum_{i=0}^{f-1} b_{i+j+1}p^i < \sum_{i=0}^{f-1} a_{i+j+1}p^i \right\}.$$

Furthermore, $d = \sum a_i p^i \in B_J$ *if and only if* $J = S$ *or* $J = \emptyset$.

Proof. First note that if $d = \sum_{i=0}^{f-1} b_i p^i \ne \sum_{i=0}^{f-1} a_i p^i$, then

$$\sum_{i=0}^{f-1} b_{i+j+1}p^i < \sum_{i=0}^{f-1} a_{i+j+1}p^i \qquad \text{if and only if} \qquad b_{j-r} < a_{j-r}$$

where $r \in \{0,\dots,f-1\}$ is chosen so that $b_{j-r} \ne a_{j-r}, b_{j-r+1} = a_{j-r+1}, \dots, b_j = a_j$. Indeed if $b_{j-r} < a_{j-r}$, then

$$\begin{aligned} b_jp^{f-1} + b_{j-1}p^{f-2} + \cdots + b_{j+1} &< b_jp^{f-1} + b_{j-1}p^{f-2} + \cdots + b_{j-r}p^{f-1-r} + p^{f-1-r} \\ &\le a_jp^{f-1} + a_{j-1}p^{f-2} + \cdots + a_{j-r}p^{f-1-r} \\ &\le a_jp^{f-1} + a_{j-1}p^{f-2} + \cdots + a_{j+1}. \end{aligned}$$

Now suppose that $d = \sum_{i=0}^{f-1} b_i p^i \in B_J$ and $j \in J$. Then $b_j \le a_j$ and if equality holds then $j-1 \in J$, so $b_{j-1} \le a_{j-1}$. Iterating, we find that either $d = \sum_{i=0}^{f-1} a_i p^i$ and $J = S$, or that

$$b_{j-r} < a_{j-r}, b_{j-r+1} = a_{j-r+1}, \dots, b_j = a_j, \text{for some } r \in \{0,\dots,f-1\},$$

yielding the desired inequality. The case $j \notin J$ is similar.

Conversely, suppose that $d \ne \sum_{i=0}^{f-1} a_i p^i$ and J is given by the formula in the statement of the lemma. If $j \in J$, then we have $b_{j-r} < a_{j-r}, b_{j-r+1} = a_{j-r+1}, \dots, b_j = a_j$, for some $r \in \{0,\dots,f-1\}$, so either $b_j < a_j$, or $b_j = a_j$ and the inequality for $j-1$ gives $j-1 \in J$. In either case we have $b_j < a_j + \delta_J(j)$. Similarly we find that $j \notin J$ implies that $a_j + \delta_J(j) \le b_j$. Finally, it is clear that $\sum_{i=0}^{f-1} a_i p^i$ is in both $B_\emptyset$ and B_S. □

Returning to the proof of Proposition 1.1, the lemma gives

$$\sum_J \left(\sum_{d \in B_J} (\tilde{y}/\tilde{x})^d \right) = (\tilde{y}/\tilde{x})^{\sum_{i=0}^{f-1} a_ip^i} + \sum_{d=0}^{q-1} (\tilde{y}/\tilde{x})^d = \begin{cases} 1+q & \text{if } \tilde{y} = \tilde{x}, \\ 1 + (\tilde{y}/\tilde{x})^{\sum_{i=0}^{f-1} a_ip^i} & \text{if } \tilde{y} \ne \tilde{x}, \end{cases}$$

from which it follows that

$$\sum_J \beta_{\vec{m}_J,\vec{n}_J}\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) = \begin{cases} (q+1)\prod_\tau \tilde{\tau}(x)^{a_\tau}, & \text{if } y = x, \\ \prod_\tau \tilde{\tau}(x)^{a_\tau} + \prod_\tau \tilde{\tau}(y)^{a_\tau}, & \text{if } y \ne x. \end{cases}$$

Now consider conjugacy classes of elements of the form $i(z)$ for $z \notin k^\times$. Choosing an embedding τ_0' of k extending τ_0 and writing $\tilde z$ for $\tilde\tau_0'(z)$, we have

$$\begin{aligned}\beta_{\vec m,\vec n}(i(z)) &= \tilde z^{\sum_{i=0}^{f-1}(q+1)m_ip^i}\sum_{\vec 0\le\vec\nu\le\vec n-\vec 1}\tilde z^{\sum_{i=0}^{f-1}(n_i-1+(q-1)\nu_i)p^i}\\ &= \tilde z^{\sum_{i=0}^{f-1}(2m_i+n_i-1)p^i}\sum_{\vec m\le\vec b\le\vec m+\vec n-\vec 1}\tilde z^{(q-1)\sum_{i=0}^{f-1}b_ip^i}.\end{aligned}$$

Summing over J and using (1) then gives

$$\begin{aligned}\sum_J\beta_{\vec m_J,\vec n_J}(i(z)) &= \tilde z^{\sum_{i=0}^{f-1}a_ip^i}\tilde z^{(q-1)(1-\delta_J(0))}\sum_J\left(\sum_{d\in B_J}\tilde z^{(q-1)d}\right)\\ &= \tilde z^{\sum_{i=0}^{f-1}a_ip^i}\left(\sum_{J\ni f-1}\left(\sum_{d\in B_J}\tilde z^{(q-1)d}\right)+\sum_{J\not\ni f-1}\left(\sum_{d\in B_J}\tilde z^{(q-1)(1+d)}\right)\right),\end{aligned}$$

where B_J is as in (2). According to Lemma 1.2, the values of $d\neq\sum_{i=0}^{f-1}a_ip^i$ contributing to the first sum are those with $0\le d<\sum_{i=0}^{f-1}a_ip^i$, the values contributing to the second are those with $\sum_{i=0}^{f-1}a_ip^i<d\le q-1$, and there is one occurrence of $\sum_{i=0}^{f-1}a_ip^i$ in each. It follows that

$$\sum_J\beta_{\vec m_J,\vec n_J}(i(z))=\tilde z^{\sum_{i=0}^{f-1}a_ip^i}\sum_{d=0}^{q}\tilde z^{(q-1)d}=0,$$

since $\tilde z^{q-1}\neq 1$, but $\tilde z^{q^2-1}=1$. This completes the proof of Proposition 1.1. □

Note that when χ is trivial, so $V\sim\det\oplus\mathrm{sp}$, the proposition gives $\overline V\sim V_{\vec 0,\vec 1}\oplus V_{\vec 0,\vec p}$, the first factor being the reduction of det and the second being that of sp. If χ is non-trivial, then $I(1,\chi)$ is irreducible and its reduction is given by the proposition.

Now we turn our attention to the $(q-1)$-dimensional representations $\Theta(\xi)$. Choosing $\tau_0' : k'\to\overline{\mathbb F}_p$ as in the proof of the theorem, we can write $\xi=(\tilde\tau_0')^n$ for some n, determined mod q^2-1. Since $\xi\neq\xi\circ\sigma$, we have that n is not divisible by $q+1$ and can therefore be written in the form $\alpha+(q+1)\beta$ with $1\le\alpha\le q$, $0\le\beta\le q-2$. Twisting by $\tilde\tau_0{}^{-\beta}\circ\det$, we can assume $n=\alpha$ and write $\xi=\tilde\tau_0'\prod_{i=0}^{f-1}(\tilde\tau_i')^{a_{\tau_i}}$ where $\tau_i'=\tau_0'\circ\mathrm{Frob}_p^i$, $\tau_i=\tau_i'|k$ and $0\le a_{\tau_i}\le p-1$ for $i=0,\dots,f-1$.

Proposition 1.3. *Let* $V=\Theta\left(\tilde\tau_0'\prod_{i=0}^{f-1}(\tilde\tau_i')^{a_{\tau_i}}\right)$ *with* $0\le a_\tau\le p-1$ *for each* $\tau\in S$. *Then* $\overline V\sim\oplus_{J\subset S}V_J$, *where* $V_J=V_{\vec m_J,\vec n_J}$ *with* $\vec m_J$ *and* $\vec n_J$ *defined as follows:*

$$m_{J,\tau} = \begin{cases}\delta_J(\tau), & \text{if } \tau=\tau_0\in J,\\ a_\tau+1, & \text{if } \tau=\tau_0\notin J,\\ 0, & \text{if } \tau\in J,\ \tau\neq\tau_0,\\ a_\tau+\delta_J(\tau), & \text{if } \tau\notin J,\ \tau\neq\tau_0,\end{cases}$$

$$\text{and}\quad n_{J,\tau} = \begin{cases}a_\tau+1-\delta_J(\tau), & \text{if } \tau=\tau_0\in J,\\ p-a_\tau-1+\delta_J(\tau), & \text{if } \tau=\tau_0\notin J,\\ a_\tau+\delta_J(\tau), & \text{if } \tau\in J,\ \tau\neq\tau_0,\\ p-a_\tau-\delta_J(\tau), & \text{if } \tau\notin J,\ \tau\neq\tau_0,\end{cases}$$

where δ_J is the characteristic function of $J^{(p)} = \{\,\tau \circ \mathrm{Frob} \,|\, \tau \in J\,\}$. Moreover the non-zero V_J are inequivalent.

Proof. Taking $(\vec{m}, \vec{n}) = (\vec{m}_J, \vec{n}_J)$ as in (1) now gives

$$\sum_{i=0}^{f-1}(2m_i + n_i - 1)p^i = 1 + (1 - \delta_J(0))(q-1) + \sum_{i=0}^{f-1} a_i p^i, \tag{3}$$

so that

$$\beta_{\vec{m}_J,\vec{n}_J}\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \tilde{x}^{1+\sum_{i=0}^{f-1} a_i p^i}\left(\sum_{d \in B'_J} (\tilde{y}/\tilde{x})^d\right),$$

where

$$B'_J = \left\{ d = \sum_{i=0}^{f-1} b'_i p^i \;\middle|\; \begin{array}{ll} \delta_J(0) \le b'_0 < a_0 + 1, & \text{if } 0 \in J, \\ a_0 + 1 \le b'_0 < p + \delta_J(0), & \text{if } 0 \notin J, \\ 0 \le b'_i < a_i + \delta_J(i), & \text{if } i \in J,\ i \ne 0, \\ a_i + \delta_J(i) \le b'_i < p, & \text{if } i \notin J,\ i \ne 0 \end{array} \right\}. \tag{4}$$

Note that if $d \in B'_J$, then $1 \le d \le q-1$. Since there are no dependence relations among the functions $w \mapsto \tilde{w}^d$ on $k^\times$ for such d, we see as in the proof of Proposition 1.1 that the non-zero V_J are inequivalent.

Lemma 1.4. *Suppose that $1 \le d \le q-1$. Write $d = \sum_{i=0}^{f-1} b_i p^i$ with $0 \le b_i \le p-1$ for each $i \in \mathbb{Z}/f\mathbb{Z}$. If $d \le \sum_{i=0}^{f-1} a_i p^i$, then*

$$d \in B'_J \iff J = \left\{ j \in \{0, \ldots, f-1\} \;\middle|\; 0 < \sum_{i=0}^{j} b_i p^i \le \sum_{i=0}^{j} a_i p^i \right\}.$$

If $\sum_{i=0}^{f-1} a_i p^i < d$, then

$$d \in B'_J \iff J = \left\{ j \in \{0, \ldots, f-1\} \;\middle|\; \sum_{i=0}^{j} b_i p^i \le \sum_{i=0}^{j} a_i p^i \right\}.$$

Proof. Write $d = \sum_{i=0}^{f-1} b'_i p^i$ with $\delta_J(0) \le b'_0 < p + \delta_J(0)$, and $0 \le b'_i < p$ for $i = 1, \ldots, f-1$. We then have $(b'_0, b'_1, \ldots, b'_{f-1}) = (b_0, b_1, \ldots, b_{f-1})$ unless $f-1 \in J$ and $b_0 = 0$, in which case

$$(b'_0, b'_1, \ldots, b'_{f-1}) = (p, p-1, \ldots, p-1, b_r - 1, b_{r+1}, \ldots, b_{f-1})$$

where r is the least positive integer such that $b_r > 0$. It follows that if $f-1 \in J$, then

$$0 < \sum_{i=0}^{j} b_i p^i \le \sum_{i=0}^{j} a_i p^i \iff \sum_{i=0}^{j} b'_i p^i \le \sum_{i=0}^{j} a_i p^i.$$

If $d \in B'_J$, then we have

$$j \in J \iff \sum_{i=0}^{j} b'_i p^i \le \sum_{i=0}^{j} a_i p^i \tag{5}$$

for $j = 0, \ldots, f-1$ by induction on j. In particular, $f-1$ is in J if and only if $d \le \sum_{i=0}^{f-1} a_i p^i$, and (5) translates into the desired formula for B'_J in either case.

Suppose conversely that J is as defined in the statement of the lemma. In particular, $f-1$ is in J if and only if $d \le \sum_{i=0}^{f-1} a_i p^i$, so (5) holds for $j = 0, \dots, f-1$. Therefore $b_0' \le a_0$ if and only if $0 \in J$, and we deduce that

$$j \in J \Longleftrightarrow b_j' < a_j + \delta_J(j)$$

for $j = 1, \dots, f-1$ by induction. It follows that $d \in B_J'$. $\square$

Returning to the proof of Proposition 1.3, the lemma gives

$$\sum_J \left(\sum_{d \in B_J'} (\tilde{y}/\tilde{x})^d \right) = \sum_{d=1}^{q-1} (\tilde{y}/\tilde{x})^d = \begin{cases} q-1 & \text{if } \tilde{y} = \tilde{x}, \\ 0 & \text{if } \tilde{y} \neq \tilde{x}, \end{cases}$$

since $(\tilde{y}/\tilde{x})^{q-1} = 1$. It follows that

$$\sum_J \beta_{\vec{m}_J, \vec{n}_J} \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) = \begin{cases} (q-1)\xi(x), & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

where $\xi = \tilde{\tau}_0' \prod_{i=0}^{f-1} (\tilde{\tau}_i')^{a_{\tau_i}}$.

Now for conjugacy classes of elements of the form $i(z)$ for $z \notin k^\times$, we have

$$\begin{aligned} \sum_J \beta_{\vec{m}_J, \vec{n}_J}(i(z)) &= \tilde{z}^{1+\sum_{i=0}^{f-1} a_i p^i} \tilde{z}^{(q-1)(1-\delta_J(0))} \sum_J \left(\sum_{d \in B_J'} \tilde{z}^{(q-1)d} \right) \\ &= \tilde{z}^{1+\sum_{i=0}^{f-1} a_i p^i} \left(\sum_{J \ni f-1} \left(\sum_{d \in B_J'} \tilde{z}^{(q-1)d} \right) + \sum_{J \not\ni f-1} \left(\sum_{d \in B_J'} \tilde{z}^{(q-1)(1+d)} \right) \right), \end{aligned}$$

where B_J' is as in (4). According to Lemma 1.4, the values of d contributing to the first sum are those with $1 \le d \le \sum_{i=0}^{f-1} a_i p^i$, the values contributing to the second are those with $\sum_{i=0}^{f-1} a_i p^i < d \le q-1$. It follows that

$$\begin{aligned} \sum_J \beta_{\vec{m}_J, \vec{n}_J}(i(z)) &= \tilde{z}^{1+\sum_{i=0}^{f-1} a_i p^i} \left(-1 - \tilde{z}^{(q-1)(1+\sum_{i=0}^{f-1} a_i p^i)} + \sum_{d=0}^{q} \tilde{z}^{(q-1)d} \right) \\ &= -\tilde{z}^{1+\sum_{i=0}^{f-1} a_i p^i} - \tilde{z}^{q(1+\sum_{i=0}^{f-1} a_i p^i)} \\ &= -\xi(z) - \xi(z^\sigma). \end{aligned}$$

This completes the proof of Proposition 1.3.

$\square$

2. THE CORRESPONDENCE

In this section we construct the bijection of Theorem 0.1. Let ω_0 and ω_0' denote fundamental characters $I_K \to \overline{\mathbb{F}}_p^\times$ corresponding to embeddings $\tau_0 : k \to \overline{\mathbb{F}}_p$ and $\tau_0' : k' \to \overline{\mathbb{F}}_p$ chosen as in the preceding section. Thus ω_0' has order $q^2 - 1$, and $\omega_0 = (\omega_0')^{q+1}$. If $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous representation, then $\rho|_{I_K}^{\mathrm{ss}}$ is equivalent to one of the form

$$\begin{array}{rl} \omega_0^r \oplus \omega_0^s & \text{for some } r, s \in \mathbb{Z}, \\ \text{or} \quad (\omega_0')^t \oplus (\omega_0')^{qt} & \text{for some } t \in \mathbb{Z} \text{ not divisible by } q+1, \end{array}$$

according to whether or not ρ is reducible.

In [5], the weight part of an analogue of Serre's conjecture is formulated over totally real fields by defining a set $W(\rho)$ of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k)$. We recall the definition in the easiest case, when $\rho|_{I_K}$ is semi-simple. In the case $\rho \sim \omega_0^r \oplus \omega_0^s$, then we define $W(\rho)$ by the rule

$$V_{\vec{m},\vec{n}} \in W(\rho) \Longleftrightarrow \left\{ \begin{array}{rcl} r & \equiv & \sum_{i=0}^{f-1} m_i p^i + \sum_{i\in J^*} n_i p^i \bmod (q-1), \\ s & \equiv & \sum_{i=0}^{f-1} m_i p^i + \sum_{i\notin J^*} n_i p^i \bmod (q-1), \end{array} \right. \text{for some } J^* \subset S$$

(where as usual, $m_i = m_{\tau_i}$ with $\tau_i = \tau_0 \circ \mathrm{Frob}^i$). In the case $\rho \sim (\omega_0')^t \oplus (\omega_0')^{qt}$ for some $c \not\equiv 0 \mod (q+1)$, we let $S' = \{0, 1, \dots, 2f-1\}$ and define $\pi : S' \to S$ by reduction mod f. We then define $W(\rho)$ by

$$V_{\vec{m},\vec{n}} \in W(\rho) \Longleftrightarrow \left\{ \begin{array}{c} t \equiv \sum_{i=0}^{f-1} (q+1) m_i p^i + \sum_{i\in J^*} n_i p^i \bmod (q^2-1) \\ \text{for some } J^* \subset S' \text{ such that } \pi : J^* \xrightarrow{\sim} S. \end{array} \right.$$

Let R_I denote the set of equivalence classes of $\overline{\mathbb{Q}}_p$-representations of I_K as above; note that there are $(q^2-q)/2$ of each type. Let R_G denote the set of equivalence classes of representations of $\mathrm{GL}_2(k)$ of the form $I(\chi_1, \chi_2)$ or $\Theta(\xi)$; note that there are $(q^2-q)/2$ of each of these as well. Recall that if V is a $\overline{\mathbb{Q}}_p$ representation of $\mathrm{GL}_2(k)$, then $\overline{V}$ denotes the semi-simplification of its reduction modulo the maximal ideal of $\overline{Z}_p$.

Theorem 2.1. *There is a bijection $\beta : R_G \to R_I$ such that if $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ restricts to $\beta(V)$ on I_K and $V_{\vec{m},\vec{n}}$ is an irreducible subrepresentation of $\overline{V}$, then $V_{\vec{m},\vec{n}} \in W(\rho)$.*

Note that Theorem 0.1 follows from Theorem 2.1 on replacing $I(\chi, \chi)$ by its q-dimensional irreducible subrepresentation and setting $\alpha(\rho) = \beta^{-1}(\rho|_{I_K}^{\mathrm{ss}})$.

Proof. We first define β for the representations considered in Propositions 1.1 and 1.3. Suppose that $b_0, \dots, b_{f-1}$ are integers with $1 \le b_i \le p$. If f is odd, then we let

$$\begin{array}{rcl} \beta(I(1, \tilde{\tau}_0^{\sum_{i=0}^{f-1}(b_i-1)p^i})) & = & (\omega_0')^{b_0+b_2p^2+\cdots+b_{f-1}p^{f-1}+b_1p^{f+1}+\cdots+b_{f-2}p^{2f-2}} \\ & & \oplus(\omega_0')^{b_1p+b_3p^3+\cdots+b_{f-2}p^{f-2}+b_0p^f+b_2p^{f+2}+\cdots+b_{f-1}p^{2f-1}} \\ \text{and } \beta(\Theta((\tilde{\tau}_0')^{1+\sum_{i=0}^{f-1}(b_i-1)p^i})) & = & \omega_0^{b_0+b_2p^2+\cdots+b_{f-1}p^{f-1}} \oplus \omega_0^{1+b_1p+b_3p^3+\cdots+b_{f-2}p^{f-2}} \end{array}$$

If f is even, then we let

$$\begin{array}{rcl} \beta(I(1, \tilde{\tau}_0^{\sum_{i=0}^{f-1}(b_i-1)p^i})) & = & \omega_0^{b_0+b_2p^2+\cdots+b_{f-2}p^{f-2}} \oplus \omega_0^{b_1p+b_3p^3+\cdots+b_{f-1}p^{f-1}} \\ \text{and } \beta(\Theta((\tilde{\tau}_0')^{1+\sum_{i=0}^{f-1}(b_i-1)p^i})) & = & (\omega_0')^{b_0+b_2p^2+\cdots+b_{f-2}p^{f-2}+p^f+b_1p^{f+1}+\cdots+b_{f-1}p^{2f-1}} \\ & & \oplus(\omega_0')^{1+b_1p+b_3p^3+\cdots+b_{f-1}p^{f-1}+b_0p^f+b_2p^{f+2}+\cdots+b_{f-2}p^{2f-2}}. \end{array}$$

There is no ambiguity in replacing $b_0 = \cdots = b_{f-1} = 1$ with $b_0 = \cdots = b_{f-1} = p$ in the formula for $\beta(I(1, \tilde{\tau}_0^{\sum_{i=0}^{f-1}(b_i-1)p^i}))$ as it just exchanges the two characters in the sum. We also need to check that the exponents of ω_0' are not divisible by $q+1$.

This follows from the fact that if f is odd, then

$$\begin{aligned}(6)\qquad & b_0 + b_2p^2 + \cdots + b_{f-1}p^{f-1} + b_1p^{f+1} + \cdots + b_{f-2}p^{2f-2} \\ & \quad \equiv b_0 - b_1p + b_2p^2 - \ldots - b_{f-2}p^{f-2} + b_{f-1}p^{f-1} \bmod q+1 \\ & \quad \text{and } 1 \le b_0 - b_1p + b_2p^2 - \ldots - b_{f-2}p^{f-2} + b_{f-1}p^{f-1} \le q,\end{aligned}$$

and if f is even, then

$$\begin{aligned}(7)\qquad & b_0 + b_2p^2 + \cdots + b_{f-2}p^{f-2} + p^f + b_1p^{f+1} + \cdots + b_{f-1}p^{2f-1} \\ & \quad \equiv -1 + b_0 - b_1p + b_2p^2 - \ldots - b_{f-2}p^{f-2} - b_{f-1}p^{f-1} \bmod q+1, \\ & \quad \text{and } 1-q \le b_0 - b_1p + b_2p^2 - \ldots - b_{f-2}p^{f-2} - b_{f-1}p^{f-1} \le 0.\end{aligned}$$

We extend β to all of R_G by twisting. If $\chi : k^\times \to \overline{\mathbb{Q}}_p^\times$ is a character, then we let $\beta(\chi)$ denote the character $I_K \to \overline{\mathbb{F}}_p^\times$ corresponding to $\overline{\chi}$ by local class field theory, i.e., if $\chi = \tilde{\tau}_0^r$, then $\beta(\chi) = \omega_0^r$. Any representation in R_G can be written in the form $(\chi \circ \det) \otimes V$ for some χ and some V for which we have already defined $\beta(V)$. We then let $\beta((\chi \circ \det) \otimes V) = \beta(\chi) \otimes \beta(V)$. We need to check there is no ambiguity in the definition. If $\chi_1 \ne \chi_2$, then $I(\chi_1, \chi_2)$ has two expressions of the above form, namely $(\chi_2 \circ \det) \otimes I(1, \chi)$ and $(\chi_2\chi \circ \det) \otimes I(1, \chi^{-1})$, so it suffices to check that $\beta(I(1,\chi)) = \beta(\chi) \otimes \beta(I(1,\chi^{-1}))$. If $\chi = \tilde{\tau}_0^{\sum_{i=0}^{f-1}(b_i-1)p^i}$ with each $b_i \in \{1, \ldots, p\}$, then $\chi^{-1} = \tilde{\tau}_0^{\sum_{i=0}^{f-1}(b'_i-1)p^i}$ where $b'_i = p+1-b_i$. It is then straightforward to check that replacing $\beta(I(1,\chi))$ with $\beta(\chi)\beta(I(1,\chi^{-1}))$ simply interchanges the two characters of I_K. Similarly each $\Theta(\xi)$ has two expressions as above, given explicitly by twisting the identity $\Theta(\xi) \sim (\chi \circ \det) \otimes \Theta(\xi')$ where $\xi = (\tilde{\tau}'_0)^{1+\sum_{i=0}^{f-1}(b_i-1)p^i}$, $\xi' = (\tilde{\tau}'_0)^{1+\sum_{i=0}^{f-1}(b'_i-1)p^i}$ and $\chi = \tilde{\tau}_0^{\sum_{i=0}^{f-1}(b_i-1)p^i}$ with each $b_i \in \{1, \ldots, p\}$ and $b'_i = p+1-b_i$. We find that $\beta(\Theta(\xi)) = \beta(\chi) \otimes \beta(\Theta(\xi'))$, the characters of I_K again being interchanged.

Since R_I and R_G have the same cardinality, it suffices to show that β is surjective in order to conclude it is a bijection. Therefore it suffices to show that every representation in R_I is a twist of one of the form $\beta(V)$ for some V as in Proposition 1.1 or 1.3. For representations of the form $(\omega'_0)^t \oplus (\omega'_0)^{qt}$, this follows from (6) and (7). Indeed since the values of $b_0 - b_1p + \cdots \pm b_{f-1}p^{f-1}$ are distinct, we see that there is an exponent in every non-zero congruence class mod $q+1$. For representations of the form $\omega^r \oplus \omega^s$, it suffices to note similarly that $r - s \bmod (q-1)$ arises as the difference of exponents of ω_0 for some $\beta(V)$.

Suppose now that $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ restricts to $\beta(V)$ on I_K. To prove the assertion about $W(\rho)$, we can twist V by $\chi \circ \det$ and ρ by a character restricting to $\beta(\chi)$ and so assume V is as in Proposition 1.1 or 1.3. We now need to show that each non-zero $V_{\vec{m}_J, \vec{n}_J}$ is in $W(\rho)$.

Suppose first that $V = I(1, \tilde{\tau}_0^{\sum_{i=0}^{f-1}(b_i-1)p^i})$ with each $b_i \in \{1, \ldots, p\}$ and f is odd. Given J, we let $J' = \{\, j \in S' \,|\, j \equiv i \bmod f \text{ for some } i \in J \,\}$, $J'_0 = \{\, j \in S' \,|\, j \text{ is even} \,\}$, $J'_1 = \{\, j \in S' \,|\, j \text{ is odd} \,\}$ and $J^* = (J'_0 \cap J') \cup (J'_1 \setminus J')$. We then have $\pi : J^* \xrightarrow{\sim} S$ and $\sum_{i=0}^{f-1}(q+1)m_{J,i}p^i + \sum_{i\in J^*} n_{J,i}p^i$ is congruent mod q^2-1 to

$$\begin{aligned}\sum_{i\notin J'}(b_i-1+\delta_J(i))p^i \quad &+ \sum_{i\in J_0'\cap J'}(b_i-1+\delta_J(i))p^i + \sum_{i\in J_1'\setminus J'}(p-b_i+1-\delta_J(i))p^i \\ &\equiv \sum_{i\in J_0'}(b_i-1+\delta_J(i))p^i + \sum_{i\in J_1'\setminus J'} p^{i+1} \\ &\equiv \sum_{i\in J_0'} b_ip^i \bmod (q^2-1)\end{aligned}$$

since $\sum_{i\in J_0'}(\delta_J(i)-1)p^i \equiv \sum_{i\in J_1'\setminus J'} p^{i+1}$. It follows that $V_{\vec{m}_J,\vec{n}_J} \in W(\rho)$.

If $V = \Theta((\tilde{\tau}_0')^{1+\sum_{i=0}^{f-1}(b_i-1)p^i})$ and f is even, then we proceed exactly as above, but with $J_0' = \{0,2,\ldots,f-2,f+1,f+3,\ldots,2f-1\}$ and $J_1' = S'\setminus J_0'$. The remaining cases are similar, but simpler. If $V = I(1,\tilde{\tau}_0^{\sum_{i=0}^{f-1}(b_i-1)p^i})$ (resp. $\Theta((\tilde{\tau}_0')^{1+\sum_{i=0}^{f-1}(b_i-1)p^i})$) and f is even (resp. odd), we let $J_0 = \{\, j\in S \,|\, j \text{ is even}\,\}$, $J_1 = \{\, j \in S \,|\, j \text{ is odd}\,\}$ and $J^* = (J_0\cap J)\cup(J_1\setminus J)$. In each case a calculation similar to the one above shows that $V_{\vec{m}_J,\vec{n}_J}\in W(\rho)$. $\square$

3. Exceptional weights

Let β be the bijection of Theorem 2.1, and suppose throughout the section that $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ restricts to $\beta(V)$ on I_K. We say that $V_{\vec{m},\vec{n}}$ is an *exceptional weight* for ρ (or V) if it lies in the complement in $W(\rho)$ of the set of constituents of $\overline{V}$. In this section we characterize the exceptional weights. We first give a sufficient condition for there to be none.

Theorem 3.1. *Suppose $V = I(\tilde{\tau}_0^c, \tilde{\tau}_0^{a+c})$ with $a = \sum_{i=0}^{f-1} a_ip^i$. If f is odd and $1\le a_i\le p-2$ for each i, then there are no exceptional weights for V. If f is even and $a \equiv \pm\frac{q-1}{p+1} \bmod (q-1)$ or $1\le a_i \le p-2$ for each i, then there are no exceptional weights for V unless $a\equiv \pm 2\frac{q-1}{p+1} \bmod (q-1)$, in which case the only exceptional weights are those $V_{\vec{m},\vec{n}}\in W(\rho)$ with $\vec{n}=\vec{p}$.*

Suppose $V = \Theta((\tilde{\tau}_0')^{1+a+c(q+1)})$ with $a=\sum_{i=0}^{f-1}a_ip^i$. If f is even and $1 \le a_i \le p-2$ for each i, then there are no exceptional weights for V. If f is odd and $1+a\equiv \pm\frac{q+1}{p+1} \bmod (q+1)$ or $1\le a_i\le p-2$ for each i, then there are no exceptional weights for V unless $1+a\equiv\pm 2\frac{q+1}{p+1} \bmod (q+1)$, in which case the only exceptional weights are those $V_{\vec{m},\vec{n}}\in W(\rho)$ with $\vec{n}=\vec{p}$.

We remark that the special cases in the statement are precisely those where ρ is the sum of two characters whose ratio is trivial or cyclotomic on inertia.

Proof. We first treat the cases where $1\le a_i\le p-2$ for each i. In this case the $V_{\vec{m}_J,\vec{n}_J}$ of Propositions 1.1 and 1.3 are all non-zero, so $\overline{V}$ has 2^f constituents. If $\beta(V)$ is irreducible, then Proposition 3.1 of [5] shows that $\#W(\rho)\le 2^f$, so it follows that equality holds and there are no exceptional weights in this case.

So suppose that $\beta(V)$ is reducible. Propositions 3.4 and 3.5 of [5] then show that $\#W(\rho)\le 2^f$ unless

$$\sum_{i=0}^{f-1}(-1)^ib_ip^i \equiv (p+1)\sum_{i\in J^*}(-1)^ip^i \bmod q-1$$

for some $J^*\subset S$, where each $b_i = a_i+1\in\{2,\ldots,p-1\}$.

If f is even, then the left-hand side is strictly between $1-q$ and 0 while the right hand side is between $-2(q-1)$ and $q-1$. Setting

$$\sum_{i=0}^{f-1}(-1)^i b_i p^i = c(q-1) + (p+1)\sum_{i\in J}(-1)^i p^i$$

for $c = 0, \pm 1$ and solving p-adically, the restriction on b_i forces either $J = \{0, 2, \ldots, f-2\}$ and $\vec{a} = (1, p-2, 1, \ldots, p-2)$ or $J = \{1, 3, \ldots, f-1\}$ and $\vec{a} = (p-2, 1, p-2, \ldots, 1)$, which gives $a \equiv \pm 2\frac{q-1}{p+1} \bmod (q-1)$. In this case one has $\#W(\rho) \le 2^f + 1$ if $p > 3$, so there is at most one exceptional weight. Note also that there is an element of $W(\rho)$ of the form $V_{\vec{m},\vec{p}}$, but no such factor of $\overline{V}$ since $a \not\equiv 0 \bmod q-1$. If $p = 3$, one has $\#W(\rho) \le 2^f + 2$ and two elements of $W(\rho)$ of the form $V_{\vec{m},\vec{p}}$ accounting for all exceptional weights.

The argument in the case of odd f is similar, but the left-hand side is strictly between 0 and $q-1$ while the right-hand side is between $-(q-1)$ and $2(q-1)$ and we get that either $J = \{0, 2, \ldots, f-1\}$ and $\vec{a} = (1, p-2, 1, \ldots, p-2, 1)$ or $J = \{1, 3, \ldots, f-2\}$ and $\vec{a} = (p-2, 1, p-2, \ldots, 1, p-2)$, giving $1 + a \equiv \pm 2\frac{q+1}{p+1} \bmod (q+1)$.

We now turn our attention to the remaining cases. Suppose first that $p > 2$ and f is even. Twisting V and ρ, we can assume

$$\vec{a} = (p-1, 0, p-1, 0, \ldots, p-1, 0) \quad \text{and} \quad \rho_{I_K} \sim \omega_0^{p+p^3+\cdots+p^{f-1}} \oplus \omega_0^{p+p^3+\cdots+p^{f-1}}.$$

For each $J^* \subset S$, we explicitly describe the $V_{\vec{m},\vec{n}}$ such that

$$\sum_{i=0}^{f/2} p^{1+2i} \equiv \sum_{i=0}^{f-1} m_i p^i + \sum_{i\in J^*} n_i p^i \equiv \sum_{i=0}^{f-1} m_i p^i + \sum_{i\notin J^*} n_i p^i \bmod (q-1).$$

Propositions 3.4 and 3.5 of [5] show that this holds for a unique $\vec{n}$ unless $J^* = \{0, 2, \ldots, f-2\}$ or $\{1, 3, \ldots, f-1\}$. For each of these two values of J^*, there are two possibilities for $\vec{n}$, namely $(p, 1, p, 1, \ldots, p, 1)$ and $(1, p, 1, p, \ldots, 1, p)$. Otherwise there is an i such that $\chi_{J^*}(i-1) = \chi_{J^*}(i)$ where χ_{J^*} is the characteristic function of J^*, and $\vec{n}$ is characterized as the unique f-tuple such that

- $n_i \in \{0, p-1, p\}$ for all i;
- if $n_{i-1} = 1$, then $n_i = p$;
- if $n_{i-1} = p-1$ or p, then $n_i = p-1$ if $\chi_{J^*}(i-1) = \chi_{J^*}(i)$;
- if $n_{i-1} = p-1$ or p, then $n_i = 1$ if $\chi_{J^*}(i-1) \ne \chi_{J^*}(i)$.

Note also that $\sum m_i p^i \bmod (q-1)$ is determined by $\vec{n}$ and J^*. It is then straightforward to check that each such $V_{\vec{m},\vec{n}}$ arises as $V_{\vec{m}_{J'},\vec{n}_{J'}}$ where

$$J' = \{\, j \in \{0, 2, \ldots, f-2\} \,|\, n_j = p-1 \text{ or } p \,\} \cup \{\, j \in \{1, 3, \ldots, f-1\} \,|\, n_j = 1 \,\},$$

so there are no exceptional weights. (Note that $(J')^*$ need not coincide with J^*.)

The case of odd f, $p > 2$ is similar. We assume

$$\vec{a} = (p-1, 0, p-1, 0, \ldots, p-1) \quad \text{and} \quad \rho_{I_K} \sim \omega_0^{1+p+p^3+\cdots+p^{f-2}} \oplus \omega_0^{1+p+p^3+\cdots+p^{f-2}}.$$

For each J^*, there is a unique possibility for $\vec{n}$ characterized exactly as in the case of f even, and we set

$$J' = \{\, j \in \{0, 2, \ldots, f-1\} \,|\, n_j = p-1 \text{ or } p \,\} \cup \{\, j \in \{1, 3, \ldots, f-2\} \,|\, n_j = 1 \,\}$$

to conclude there are no exceptional weights.

Finally if $p = 2$, then one argues as above using Proposition 3.6 of [5], but with two changes. Firstly, we find also that there are two possibilities for $\vec{n}$ if $J^* = \emptyset$ or S, namely $\vec{n} = (1, 1, \ldots, 1)$ or $(2, 2, \ldots, 2)$, the latter being exceptional. Secondly, to generalize the characterization of $\vec{n}$ one defines $n_i(x) \in \{0, x-1, x\}$ and then sets $\vec{n} = \vec{n}(p)$ with $p = 2$. $\square$

We finish with a complete characterization of the exceptional weights.

Theorem 3.2. *Suppose that* $V_{\vec{m},\vec{n}} \in W(\rho)$.

If $\rho|_{I_K} \sim \omega_0^r \oplus \omega_0^s$, *then* $V_{\vec{m},\vec{n}}$ *is exceptional if and only if for each* $J^* \subset S$ *such that*

$$r \equiv \sum_{i=0}^{f-1} m_i p^i + \sum_{i \in J^*} n_i p^i, \quad s \equiv \sum_{i=0}^{f-1} m_i p^i + \sum_{i \notin J^*} n_i p^i \bmod (q-1), \tag{8}$$

we have $n_i = p$ *and* $\chi_{J^*}(i-1) = \chi_{J^*}(i)$ *for some* $i \in S$.

If $\rho|_{I_K} \sim (\omega_0')^t \oplus (\omega_0')^{qt}$, *then* $V_{\vec{m},\vec{n}}$ *is exceptional if and only if for each* $J^* \subset S'$ *such that*

$$t \equiv \sum_{i=0}^{f-1} (q+1) m_i p^i + \sum_{i \in J^*} n_i p^i \bmod (q^2-1) \quad \textit{and} \quad \pi : J^* \xrightarrow{\sim} S, \tag{9}$$

we have $n_i = p$ *and* $\chi_{J^*}(i-1) = \chi_{J^*}(i)$ *for some* $i \in S'$.

Proof. We note first that every $V_{\vec{m},\vec{n}}$ as in the statement of the theorem is indeed exceptional, for if it is equivalent to $V_{\vec{m}_J,\vec{n}_J}$ for some $J \subset S$, then the proof of Theorem 2.1 provides a J^* such that (8) or (9) holds, but the explicit formula for $\vec{n}_J$ shows that $n_i < p$ whenever $\chi_{J^*}(i-1) = \chi_{J^*}(i)$.

Suppose on the other hand that (8) or (9) holds for some J^* such that $\chi_{J^*}(i-1) \neq \chi_{J^*}(i)$ whenever $n_i = p$. We then choose J so that J^* is as in the proof of Theorem 2.1 and verify that $V_{\vec{m},\vec{n}} = V_{\vec{m}_J,\vec{n}_J}$ (except possibly in the case of reducible ρ with $r = s$, where the result is already immediate from Theorem 3.1). We can twist V and ρ and so assume V is as in the statements of Proposition 1.1 or 1.3.

Suppose first that $V = I\left(1, \tilde{\tau}_0^{\sum_{i=0}^{f-1} a_i p^i}\right)$ with f even and $0 \le a_i p - 1$ for each i. We then have

$$\sum_{i=0}^{f-1} m_i p^i + \sum_{i \in J^*} n_i p^i \equiv b_0 + b_2 p^2 + \cdots b_{f-2} p^{f-2} \bmod (q-1) \tag{10}$$

$$\sum_{i=0}^{f-1} m_i p^i + \sum_{i \notin J^*} n_i p^i \equiv b_1 p + b_3 p^3 + \cdots b_{f-1} p^{f-1} \bmod (q-1) \tag{11}$$

where each $b_i = a_i + 1$. Let $b_i' = n_i - \delta_J(i) + 1$ for $i \in J$ and $b_i' = p - n_i - \delta_J(i) + 1$ for $i \notin J$. The condition that $\chi_{J^*}(i-1) \neq \chi_{J^*}(i)$ whenever $n_i = p$ guarantees that $1 \le b_i' \le p$ for each i. It is then straightforward to check that

$$\sum_{i=0}^{f-1} (-1)^i b_i' p^i \equiv \sum_{i \in J^*} n_i p^i - \sum_{i \notin J^*} n_i p^i \bmod (q-1),$$

which by (10) is congruent to $\sum_{i=0}^{f-1} b_i p^i$. Since $1 \le b_i, b_i' \le p$ for each i and we have ruled out the case $\sum_{i=0}^{f-1} b_i p^i \equiv 0 \bmod (q-1)$, it follows that $b_i = b_i'$ for all i, and therefore that $\vec{n} = \vec{n}_J$. We then compute that

$$\begin{aligned}\sum_{i=0}^{f-1} m_i p^i &\equiv \sum_{\text{even } i} b_i p^i - \sum_{i \in J^*} n_i p^i \\ &\equiv \sum_{\text{even } i \notin J^*} (a_i + \delta_J(i)) p^i + \sum_{\text{odd } i \in J^*} (a_i + \delta_J(i)) p^i \\ &\equiv \sum_{i \in S} m_{J,i} p^i,\end{aligned}$$

hence $V_{\vec{m},\vec{n}} = V_{\vec{m}_J,\vec{n}_J}$.

Suppose next that $V = I\left(1, \tilde{\tau}_0^{\sum_{i=0}^{f-1} a_i p^i}\right)$ but f is odd. We then start with the congruence

$$\sum_{i=0}^{f-1} m_i (q+1) p^i + \sum_{i \in J^*} n_i p^i \equiv b_0 + b_2 p^2 + \cdots b_{2f-2} p^{2f-2} \bmod (q^2-1)$$

instead of (10). Defining b_i' and arguing as above then gives

$$\sum_{i=0}^{f-1} (-1)^i b_i' p^i \equiv \sum_{i=0}^{f-1} (-1)^i b_i p^i \bmod (q+1),$$

so $\vec{b} = \vec{b}'$ and $\vec{n} = \vec{n}_J$. Similarly one finds that

$$(q+1) \sum_{i=0}^{f-1} m_i p^i \equiv \sum_{\text{even } i \in S'} b_i p^i - \sum_{i \in J^*} n_i p^i \equiv (q+1) \sum_{i \notin J} (a_i + \delta_J(i)) p^i \bmod (q^2-1)$$

giving $V_{\vec{m},\vec{n}} = V_{\vec{m}_J,\vec{n}_J}$.

Now suppose that $V = \Theta\left((\tilde{\tau}_0')^{1+\sum_{i=0}^{f-1} a_i p^1}\right)$ with f even. We then have

$$\begin{aligned}&\textstyle\sum_{i=0}^{f-1} m_i (q+1) p^i + \sum_{i \in J^*} n_i p^i \\ &\qquad \equiv b_0 + b_2 p^2 + \cdots + b_{f-2} p^{f-2} + p^f + b_{f+1} p^{f+1} + \cdots + b_{2f-1} p^{2f-1} \bmod (q^2-1).\end{aligned}$$

We define b_i' for $i > 0$ as above, but set $b_0' = n_0 + \delta_J(0)$ or $p - n_0 + \delta_J(0)$ according to whether $0 \in J^*$. Arguing as above, with special attention to the terms with $i = 0, f$, then gives

$$\sum_{i=0}^{f-1} (-1)^i b_i' p^i \equiv 1 + \sum_{i \in J^*, 0 \le i < f} n_i p^i - \sum_{i \notin J^*, 0 \le i < f} n_i p^i \equiv \sum_{i=0}^{f-1} (-1)^i b_i p^i \bmod (q+1),$$

so that $\vec{b} = \vec{b}'$ and $\vec{n} = \vec{n}_J$. Again computing mod $q^2 - 1$, but with special attention to $i = 0, f$, gives $\sum_{i=0}^{f-1} m_i p^i \equiv \sum_{i=0}^{f-1} m_{J,i} p^i \bmod (q-1)$ so that $V_{\vec{m},\vec{n}} = V_{\vec{m}_J,\vec{n}_J}$.

Finally suppose that $V = \Theta\left((\tilde{\tau}_0')^{1+\sum_{i=0}^{f-1} a_i p^1}\right)$ with f odd. Starting with

$$\begin{aligned}\sum_{i=0}^{f-1} m_i p^i + \sum_{i \in J^*} n_i p^i &\equiv b_0 + b_2 p^2 + \cdots b_{f-1} p^{f-1} \bmod (q-1) \\ \sum_{i=0}^{f-1} m_i p^i + \sum_{i \notin J^*} n_i p^i &\equiv 1 + b_1 p + b_3 p^3 + \cdots b_{f-1} p^{f-1} \bmod (q-1),\end{aligned}$$

and defining b'_i as in the preceding case, we get

$$\sum_{i=0}^{f-1}(-1)^i b'_i p^i \equiv \sum_{i=0}^{f-1}(-1)^i b_i p^i \bmod (q-1),$$

again giving $\vec{b} = \vec{b}'$ since $r \neq s$. Checking again that $\sum_{i=0}^{f-1} m_i p^i \equiv \sum_{i=0}^{f-1} m_{J,i} p^i \bmod (q-1)$ yields $V_{\vec{m},\vec{n}} = V_{\vec{m}_J,\vec{n}_J}$. $\square$

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